CHAPTER 4

Extension

Solving Systems Using Augmented Matrices

***What** you should learn*

GOAL Solve systems of linear equations using elementary row operations on augmented matrices.

***Why** you should learn it*

The use of augmented matrices allows you to solve a linear system by suppressing the variables and working only with the coefficients and constants.

GOAL USING ELEMENTARY ROW OPERATIONS

A matrix containing the coefficient matrix and the matrix of constants for a system of linear equations is called the **augmented matrix** of the system.

LINEAR SYSTEM

$$x - 2y = 7$$
$$-3x + 5y = -4$$

AUGMENTED MATRIX

$$\left[\begin{array}{cc:c} 1 & -2 & 7 \\ -3 & 5 & -4 \end{array}\right]$$

Recall from Chapter 3 that equations in a system can be multiplied by a constant, or a multiple of one equation can be added to another equation. Similar operations can be performed on the rows of an augmented matrix to solve the corresponding system.

ELEMENTARY ROW OPERATIONS

Two augmented matrices are *row-equivalent* if their corresponding systems have the same solution(s). Any of these row operations performed on an augmented matrix will produce a matrix that is row-equivalent to the original:

- Interchange two rows.
- Multiply a row by a nonzero constant.
- Add a multiple of one row to another row.

To solve a system, use elementary row operations to transform the original augmented matrix into a matrix having 1's along the main diagonal and 0's below the main diagonal. A matrix of this form is said to be in *triangular form.*

EXAMPLE 1 *Using Row Operations to Solve a Two-Variable System*

LINEAR SYSTEM

$$x - 2y = 7$$
$$-3x + 5y = -4$$

AUGMENTED MATRIX

$$\left[\begin{array}{cc:c} 1 & -2 & 7 \\ -3 & 5 & -4 \end{array}\right]$$

Add 3 times the first equation to the second equation. You get this system:

$$x - 2y = 7$$
$$-y = 17$$

Add 3 times the first row, R_1, to the second row, R_2.

$$3R_1 + R_2 \rightarrow \left[\begin{array}{cc:c} 1 & -2 & 7 \\ 0 & -1 & 17 \end{array}\right]$$

Multiply the second equation by -1.

$$x - 2y = 7$$
$$y = -17$$

Multiply the second row by -1.

$$(-1)R_2 \rightarrow \left[\begin{array}{cc:c} 1 & -2 & 7 \\ 0 & 1 & -17 \end{array}\right]$$

The second row of the matrix tells you that $y = -17$. Substitute -17 for y in the equation for the first row: $x - 2(\mathbf{-17}) = 7$, or $x = -27$. The solution is $(-27, -17)$.

EXAMPLE 2 *Using Row Operations to Solve a Three-Variable System*

LINEAR SYSTEM

$$\begin{aligned} 2x + 4y + 5z &= 5 \\ x + 3y + 3z &= 2 \\ 2x + 4y + 6z &= 2 \end{aligned}$$

Add -1 times the first equation to the third equation. You get this system:

$$\begin{aligned} 2x + 4y + 5z &= 5 \\ x + 3y + 3z &= 2 \\ z &= -3 \end{aligned}$$

Add -0.5 times the first equation to the second equation.

$$\begin{aligned} 2x + 4y + 5z &= 5 \\ y + 0.5z &= -0.5 \\ z &= -3 \end{aligned}$$

Multiply the first equation by 0.5.

$$\begin{aligned} x + 2y + 2.5z &= 2.5 \\ y + 0.5z &= -0.5 \\ z &= -3 \end{aligned}$$

AUGMENTED MATRIX

$$\left[\begin{array}{ccc:c} 2 & 4 & 5 & 5 \\ 1 & 3 & 3 & 2 \\ 2 & 4 & 6 & 2 \end{array}\right]$$

Add -1 times the first row to the third row.

$$(-1)R_1 + R_3 \rightarrow \left[\begin{array}{ccc:c} 2 & 4 & 5 & 5 \\ 1 & 3 & 3 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

Add -0.5 times the first row to the second row.

$$-0.5R_1 + R_2 \rightarrow \left[\begin{array}{ccc:c} 2 & 4 & 5 & 5 \\ 0 & 1 & 0.5 & -0.5 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

Multiply the first row by 0.5.

$$0.5R_1 \rightarrow \left[\begin{array}{ccc:c} 1 & 2 & 2.5 & 2.5 \\ 0 & 1 & 0.5 & -0.5 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

The third row of the matrix tells you that $z = -3$. Substitute -3 for z in the equation for the second row, $y + 0.5z = -0.5$, to obtain $y + 0.5(-3) = -0.5$, or $y = 1$. Then substitute -3 for z and 1 for y in the equation for the first row, $x + 2y + 2.5z = 2.5$, to obtain $x + 2(1) + 2.5(-3) = 2.5$, or $x = 8$. The solution is $(8, 1, -3)$.

EXERCISES

SOLVING SYSTEMS **Use an augmented matrix to solve the linear system.**

1. $6x + 4y = 8$
$3x + 3y = 9$

2. $x + y = 2$
$7x + 8y = 21$

3. $x + 2y = -9$
$-2x - 3y = 14$

4. $x - 3y = 5$
$-2x - 4y = 20$

5. $3x + 2y = 2$
$5x - 6y = 50$

6. $x + y = -1$
$7x + 9y = -19$

7. $-2x - y = -5$
$6x + 5y = 17$

8. $9x - 4y = 2$
$-6x - 16y = -6$

9. $-12x + 15y = 3$
$-7x - 20y = -4$

10. $2x + 6y + 3z = 2$
$x + 3y + z = 1$
$x + 5y + 2z = -1$

11. $2x + 6y + 3z = 8$
$x + 5y + 5z = 1$
$x + 3y + z = 3$

12. $2x + 10y = 28$
$x + 3y + 4z = 22$
$x + 5y - z = 10$

13. $x + 4y - 2z = 3$
$x + 3y + 7z = 1$
$2x + 9y + z = 8$

14. $x - y + 3z = 6$
$x - 2y = 5$
$2x - 2y + 5z = 9$

15. $x + 2z = 4$
$x + y + z = 6$
$3x + 3y + 4z = 28$

16. **CRITICAL THINKING** Try using an augmented matrix to solve the given system. What happens? What can you say about the system's solution(s)?

$$\begin{aligned} x - 2y + 7z &= 6 \\ 5x - 10y + 35z &= 30 \\ 3x - 6y + 21z &= 18 \end{aligned}$$